

Non-linear properties of strongly pumped lasers

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Bloch equations for the atomic population and the polarization/coherence and the equation of motion for the photon number in a laser are solved in steady state as a function of the pump rate. Two level atom and two modes of three levels atom are investigated. Close to threshold the usual linear dependence of the intensity on the pump rate is found for all cases. However, far above threshold strongly nonlinear dependence is encountered. In the cases for which the pump connects the lower lasing state to one of the excited states the character of the non-linearity differs crucially from the cases when the pump is not related directly to the lower lasing state. Non-monotonic dependence of laser intensity upon the pump rate is predicted. Detailed discussion of the nonlinear behavior is presented, including saturation and depletion effects.

The behavior of the number of photons and the laser intensity as function of the pump rate near and above threshold has recently been studied [1]. In particular, linear dependence of the intensity close to threshold and some non-linearity above threshold has been found [2, 3]. The purpose of this paper is to point out the non-linearity of the laser intensity as function of pump and the crucial role of the rate of depletion of the lower lasing state in the dynamics of lasers. It is usually believed [4] that the faster is the rate of depletion of the lower laser level, the better, because this gives rise to a larger inversion at the lasing transition. However, as it is shown in this paper, there is a limitation on the depletion rate of the lower lasing level. For any arbitrarily large pump rate there exists a critical value of the depletion rate of the lower lasing level, at which the lasing is broken down.

In this paper we study closed two and three level systems with incoherent pump. Consider first two level scheme shown on Fig. 1. The corresponding optical Bloch equations are

$$\dot{\rho}_{11} = -\gamma\rho_{11} + \Gamma\rho_{00} - ig(z\rho_{01} - c.c.), \quad (1)$$

$$\dot{\rho}_{10} = -\gamma_{\perp}\rho_{10} + igz(\rho_{11} - \rho_{00}), \quad (2)$$

$$\dot{n} = -2\kappa n + iNg(z\rho_{01} - c.c.), \quad (3)$$

where N is the total number of atoms, g is the electric dipole coupling, κ is cavity decay rate, $n = z^*z$ is the number of photons inside the cavity, γ and Γ are spontaneous emission rate and the pump rate respectively. The steady state solution for the photon number n reads

$$n = \frac{N\gamma}{4\kappa} [P - 1 - (P + 1) \frac{\kappa\gamma_{\perp}}{Ng^2}], \quad (4)$$

where $P \equiv \Gamma/\gamma$ is the relative pump rate. It follows from eq. (4) that in order to get lasing the following condition must be satisfied

$$P > P_{thr} = \frac{1 + \frac{\kappa\gamma_{\perp}}{Ng^2}}{1 - \frac{\kappa\gamma_{\perp}}{Ng^2}}, \quad (5)$$

which is usually referred to as a threshold condition. The necessary condition for eq. (5) to be true is

$$\gamma_{\perp} < \frac{Ng^2}{\kappa}. \quad (6)$$

If this inequality is not satisfied, lasing cannot be obtained at any pump rate. In case of $Ng^2/\kappa\gamma_{\perp} \gg 1$, when eq. (6) is satisfied automatically, the threshold condition is simple

$$P > 1 \quad (7)$$

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which means that the pump rate Γ should exceed the decay rate γ of the upper lasing level. Inequality (6) gives rise to a restriction on the pump rate. Indeed, it follows from the derivation of the relaxation rates (see Refs. [5]) that

$$\gamma_{\perp} = \frac{\Gamma + \gamma + \gamma_{ph}}{2} \quad (8)$$

where γ_{ph} stands for the contribution from the phase destruction processes like elastic collisions. Combining eqs. (6)- (8) we obtain the following restriction for the pump rate P

$$1 < P < \frac{2Ng^2}{\kappa\gamma} - 1 - \delta, \quad \delta \equiv \frac{\gamma_{ph}}{\gamma}. \quad (9)$$

At first glance at eq. (4), the dependence of the mean photon number n upon the pump parameter P seems to be linear. However this is not true due to the dependence of the transversal relaxation rate γ_{\perp} upon the pump parameter. Indeed, taking into account eq. (8) we get, instead of eq. (4), the following expression for the mean photon number

$$n = \frac{N\gamma}{4\kappa} [P - 1 - (P + 1)(P + 1 + \delta) \frac{\kappa\gamma}{2Ng^2}]. \quad (10)$$

The dependence (10) is not linear, it is rather quadratic in P (see Fig. 2) with maximum at

$$P_0 = \frac{Ng^2}{\kappa\gamma} - 1 - \frac{\delta}{2}. \quad (11)$$

Note that condition (9) is necessary but not enough to get lasing. The region of the pump parameter P in which lasing is really possible is given by (see fig. 2)

$$1 < P < \frac{2Ng^2}{\kappa\gamma} - \delta - 3, \quad (12)$$

Here we assumed that $Ng^2/\kappa\gamma \gg 1$ which is normally true. Maximum value of the photon number is defined by eqs. (10), (11) and has the following form, provided $Ng^2/\kappa\gamma \gg 1$

$$n_{max} \approx \frac{1}{8} \left(\frac{Ng}{\kappa} \right)^2 \quad (13)$$

The non-linearity in eq. (10) stems from the fact that according to eq. (6) the rate of transversal relaxation γ_{\perp} depends upon the pump rate Γ . The larger is the pump rate, the shorter is atomic radiative life time which is inverse proportional to γ_{perp} . Thus, a "too fast" pump can give rise to a reduction of the photon number (or intensity) and can even switch off the lasing process. At relatively low pump this reduction is compensated by the gain which is given by the linear term in eq. (10). However, when the pump parameter P becomes large enough, i.e. of order of $P \sim Ng^2/\kappa\gamma$, the non-linearity plays an important role and cannot be neglected. The two level model exhibits an unusual non-linearity with respect to pump rate.

To get a more realistic model one has to involve multilevel schemes. In such schemes however, the depletion of the lower lasing level is not necessarily associated with pumping mechanism and therefore, the character of non-linearity may be quite different.

Let us consider now a more realistic three level model shown in fig. 3, which is a particular case of a general three level model considered in Ref. [6]. Note that the two schemes depicted in figs. 3a and 3b are equivalent mathematically, i.e., they are formally described by the same set of equations. However, physically the dynamic properties of these two schemes are completely different because of the different ways of pumping as described by the role of the relevant parameters (see figs. 3a and 3b). In the scheme of fig. 3a the lower lasing state $|0\rangle$ is depleted with the rate γ_{02} to the ground state $|2\rangle$, and then the pumping is used to excite the atom to the upper lasing state $|1\rangle$ with pump rate γ_{21} . In the scheme of fig. 3b the pump mechanism is used to excite the atom from the lower lasing state $|0\rangle$ which is now the ground state, to the upper lasing state $|2\rangle$ which is then depleted with rate γ_{21} . In the last case the role of the pump rate is played by the rate γ_{02} . As will be seen in the following this difference in the pump mechanism is crucial for the dynamics of laser.

Let us write down a set of optical Bloch equations for the three level model (fig. 3).

$$\dot{\rho}_{11} = \gamma_{21}\rho_{22} - \gamma_{10}\rho_{11} - g(iz\rho_{01} + c.c.), \quad (14)$$

$$\dot{\rho}_{22} = \gamma_{02}\rho_{00} - \gamma_{21}\rho_{22}, \quad (15)$$

$$\dot{\rho}_{10} = -\gamma_{\perp}\rho_{10} + igz(\rho_{11} - \rho_{00}), \quad (16)$$

$$\dot{n} = -2\kappa n + Ng(iz\rho_{01} + c.c.) \quad (17)$$

The steady state solution for the number of photons inside the cavity reads

$$n = \frac{N}{2\kappa} \frac{\gamma_{21}(\gamma_{02} - \gamma_{10})}{\gamma_{02} + 2\gamma_{21}} - \frac{\gamma_{\perp}}{2g^2} \frac{\gamma_{02}\gamma_{21} + \gamma_{02}\gamma_{10} + \gamma_{21}\gamma_{10}}{\gamma_{02} + 2\gamma_{21}}. \quad (18)$$

This formulae is valid for both models shown in figs. 3a and 3b. It seems from eq. (18) that the photon number n depends upon both rates γ_{02} and γ_{21} in a similar way, i.e. both dependencies are of linear-fractional type, which means that n tends asymptotically to some fix value when increasing either the rate γ_{02} or γ_{21} . This type of non-linearity has been brought about by the nonlinear dependence of both longitudinal relaxation rate and equilibrium inversion Δ upon the rates γ_{02} and γ_{21} . This can be shown explicitly by rewriting eq. (18) in the following form

$$n = \frac{N\gamma_{\parallel}\Delta}{4\kappa} - \frac{\gamma_{\perp}\gamma_{\parallel}}{4g^2}, \quad (19)$$

where

$$\gamma_{\parallel} \equiv \frac{2(\gamma_{21}\gamma_{02} + \gamma_{02}\gamma_{10} + \gamma_{21}\gamma_{21})}{\gamma_{02} + 2\gamma_{21}}, \quad (20)$$

$$\Delta \equiv \frac{\gamma_{21}(\gamma_{02} - \gamma_{10})}{\gamma_{21}\gamma_{02} + \gamma_{02}\gamma_{10} + \gamma_{21}\gamma_{21}}. \quad (21)$$

However, as it has already been mentioned in the discussion of the two level model, the dipole relaxation rate γ_{\perp} is related to the other rates. In the three level case under consideration the following relation for γ_{\perp} takes place

$$\gamma_{\perp} = \frac{1}{2}(\gamma_{10} + \gamma_{02} + \gamma_{ph}), \quad (22)$$

where γ_{ph} stands for collisional dephasing rate. Note that γ_{\perp} depends on γ_{02} but does not depend upon the rate γ_{21} . It is this asymmetry that is responsible for the difference in dynamic properties of lasers with the above two different pump mechanisms. Indeed, in the case of the scheme of fig. 3a, γ_{\perp} does not depend upon the pump rate γ_{21} and hence the dependence of the photon number upon pump remains linear-fractional. In contrast, in the case of fig. 3b the role of pump rate plays γ_{02} and thus γ_{\perp} brings about additional dependence upon the pump rate. Therefore the type of non-linearity of the photon number as function of pump in this case differs from that of the scheme 3a. This difference is crucial for the dynamics of the lasers with the different pump schemes.

Let us consider the two above mentioned modes of pumping separately. In the case of the scheme on fig. 3a equations (18) and (22) result in the following expression for the photon number

$$n_1 = \lambda_1 \frac{P_1(1 - \epsilon_1) - s_1(1 + \epsilon_1 + \delta_1)[P_1(1 + \epsilon_1) + \epsilon_1]}{1 + 2P_1} \quad (23)$$

where

$$\lambda_1 \equiv \frac{N\gamma_{02}}{2\kappa}, \quad s_1 \equiv \frac{\kappa\gamma_{02}}{2Ng^2}, \quad \epsilon_1 \equiv \frac{\gamma_{10}}{\gamma_{02}}, \quad P_1 \equiv \frac{\gamma_{21}}{\gamma_{02}}, \quad \delta_1 \equiv \frac{\gamma_{ph}}{\gamma_{02}}. \quad (24)$$

In fig. 4a n_1 is plotted as function of pump parameter P_1 . At the beginning, when $P_1 \ll 1$, n_1 grows linearly with the pump rate P_1 . However, when the pump is strong enough, i.e. $\gamma_{21} \geq \frac{\gamma_{02}}{2}$, the dependence $n_1(P_1)$ becomes non-linear. The origin of this non-linearity lies in the large disparity between the rates of depletion of the states $|2\rangle$ and $|0\rangle$. When the lower lasing state $|2\rangle$ is depleted much faster than the ground state $|0\rangle$, there is a bottleneck at the transition $|0\rangle \rightarrow |2\rangle$ so that further increase of the pump rate does not result in increase of the photon number. This kind of non-linearity was mentioned in Ref.[3].

Threshold condition for the scheme 3a is

$$P_1 > \frac{\epsilon_1 s_1 (1 + \epsilon_1 + \delta_1)}{1 - \epsilon_1 - s_1 (1 + \epsilon_1 + \delta_1) (1 + \epsilon_1)} \quad (25)$$

It follows from eq.(25) that there are two additional necessary conditions for getting lasing:

$$\epsilon_1 < 1, \quad (26)$$

$$s_1 < \frac{1 - \epsilon_1}{(1 + \epsilon_1 + \delta_1)(1 + \epsilon_1)}. \quad (27)$$

As it follows from the definition of the equilibrium inversion, Δ , (see eq.(21)), the condition (26) merely means that there should be initial positive inversion at the lasing transition. As for eq.(27) we can rewrite it assuming that $\frac{\kappa\gamma_{10}}{2Ng^2} \ll 1$ in the following form

$$1 < \frac{\gamma_{02}}{\gamma_{10}} < \frac{2Ng^2}{\kappa\gamma_{10}} - \frac{\gamma_{ph}}{\gamma_{10}} - 3 \quad (28)$$

which is similar to eq.(12). We came to the same restriction for the rate of depletion of the lower lasing level as for the two level model. This time however, this does not result in any restriction for the pump rate γ_{21} , which can be readily understood because in contrast to the two level scheme, in the scheme under consideration, there is no connection between the transversal relaxation rate and the pump rate. Inequality (27) has also another important meaning which becomes obvious if we rewrite it in the following way

$$N > N_{min} = \frac{\kappa\gamma_{02}}{2g^2} \frac{(1 + \epsilon_1 + \delta_1)(1 + \epsilon_1)}{1 - \epsilon_1}. \quad (29)$$

Thus there is a minimal number of atoms N_{min} needed to get lasing. If $N \leq N_{min}$ one cannot get lasing at any pump rate.

Let us consider the scheme shown in fig. 3b. Now instead of eq.(23) we have for the photon number

$$n_2 = \lambda_2 \frac{P_2 - \epsilon_2 - s_2(P_2 + \epsilon_2 + \delta_2)(P_2 + \epsilon_2 + P_2\epsilon_2)}{P_2 + 2}, \quad (30)$$

where

$$\lambda_2 \equiv \frac{N\gamma_{21}}{2\kappa}, \quad s_2 \equiv \frac{\kappa\gamma_{21}}{2Ng^2}, \quad \epsilon_2 \equiv \frac{\gamma_{10}}{\gamma_{21}}, \quad P_2 \equiv \frac{\gamma_{02}}{\gamma_{21}}, \quad \delta_2 \equiv \frac{\gamma_{ph}}{\gamma_{21}}. \quad (31)$$

Note that the third term in the numerator contains additional dependence upon the relative pump rate P_2 . This additional dependence has been brought about by the fact that the dipole relaxation rate γ_{\perp} depends on the pump rate γ_{02} . The photon number n_2 is plotted in fig. 4b as a function of the pump rate. Again, as in the previous case, the photon number grows linearly at relatively weak pump and becomes non-linear at strong pump. However, the character of this non-linearity is crucially different. It has the same form as in the case of the two level scheme (cf. fig. 2). This difference is due to the above mentioned additional dependence of the photon number upon the pump. The photon number has a maximum at

$$P_2 = \frac{Ng^2}{\kappa\gamma_{21}} - \frac{\delta_2}{2} - \epsilon_2. \quad (32)$$

Here we assumed that $\epsilon_2 \ll 1$. Eq.(32) is similar to eq.(11) for the two level model. In order to get lasing the following double inequality must be satisfied, provided $s_2, \epsilon_2 \ll 1$

$$1 < P_2 < \frac{1}{s_2} - \delta_2 \quad (33)$$

which is similar to eq.(12) for the two level model. Again we got the same restriction for the rate of depletion of the lower lasing state $|0\rangle$ as in the case of eq.(28). This time however, this is the restriction for the pump rate P_2 since when P_2 reaches the value of $1/s_2 - \delta_2$, the lasing is broken down for the same reason as in the case of two level

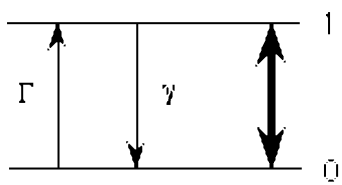
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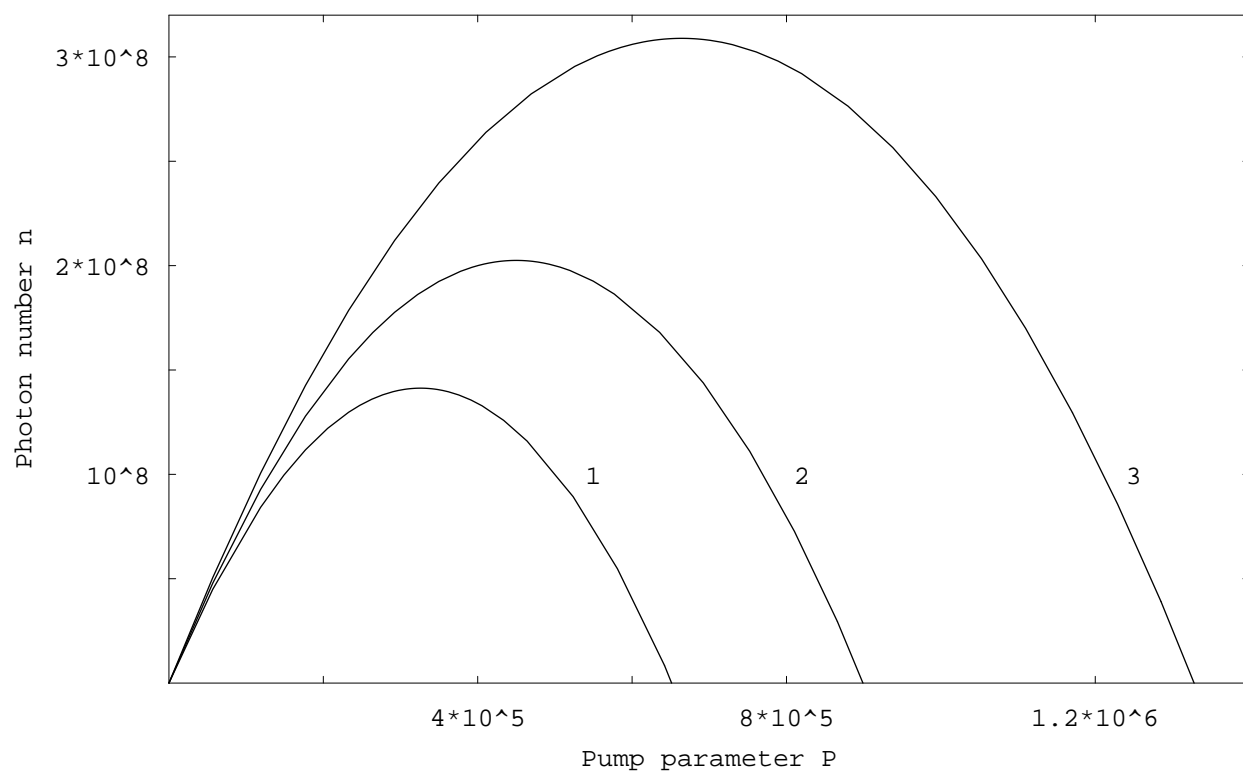
In summary, we have shown that the nonlinear dependence of the photon number upon the pump parameter is very sensitive to the type of pump mechanism. When the pumping is used to excite the lower lasing state to one of the upper atomic states, the laser exhibits a peculiar behavior shown in figs. 2 and 4b for two and three level schemes respectively. The number of photons as a function of pump reaches its maximum and then slows down until some critical point at which the lasing ceases. It should be noted that the described non-linearity manifests itself at very large pump rates when the ratio p/p_{thr} reaches a few orders of magnitude. Such a regime may be difficult to achieve in conventional lasers. For instance, in order to reach the non-linear region in experiments with GaAs laser used in Ref. [7] the ratio p/p_{thr} should be of order of 10^5 . Nevertheless, it may be realized in a microlaser.

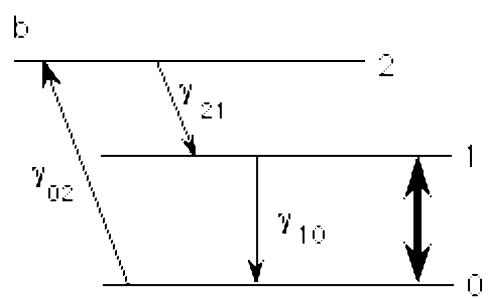
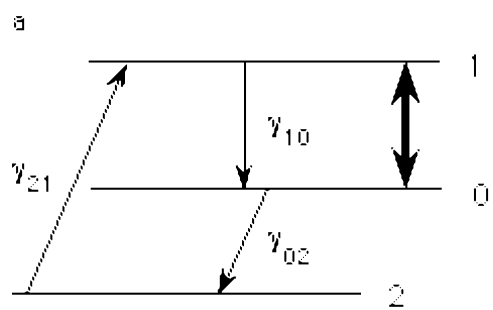
This work was supported in part by the grant from the Israeli Ministry of Immigrant Absorption.

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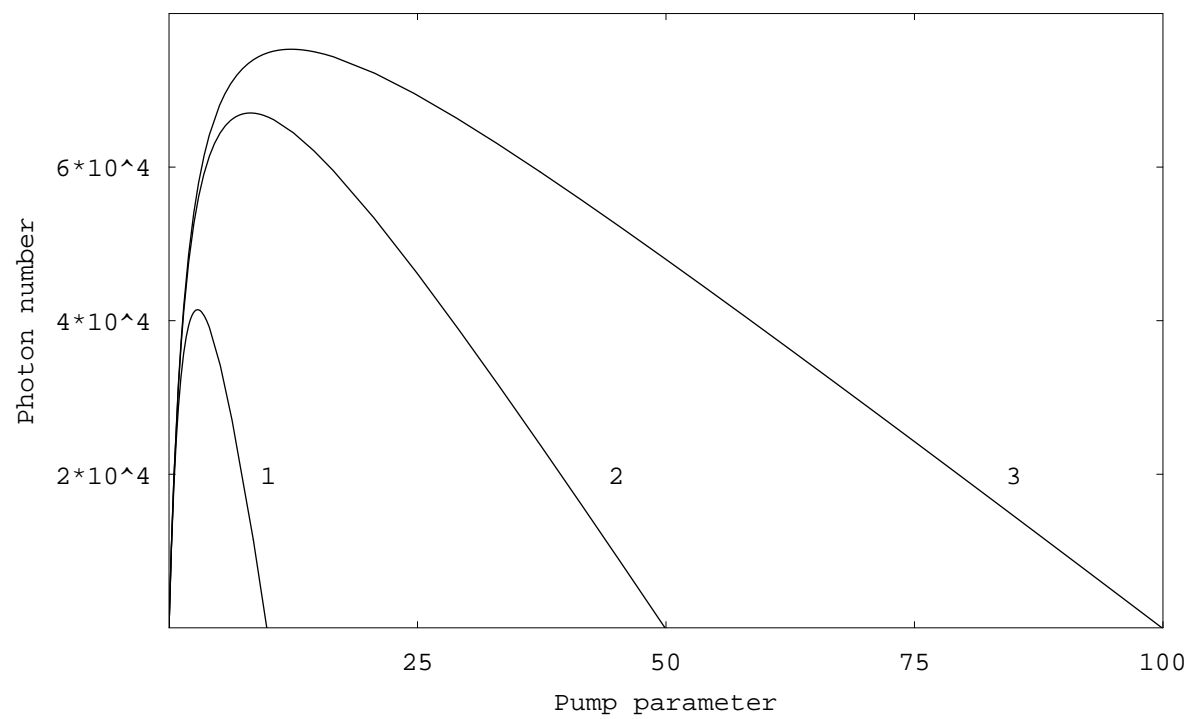
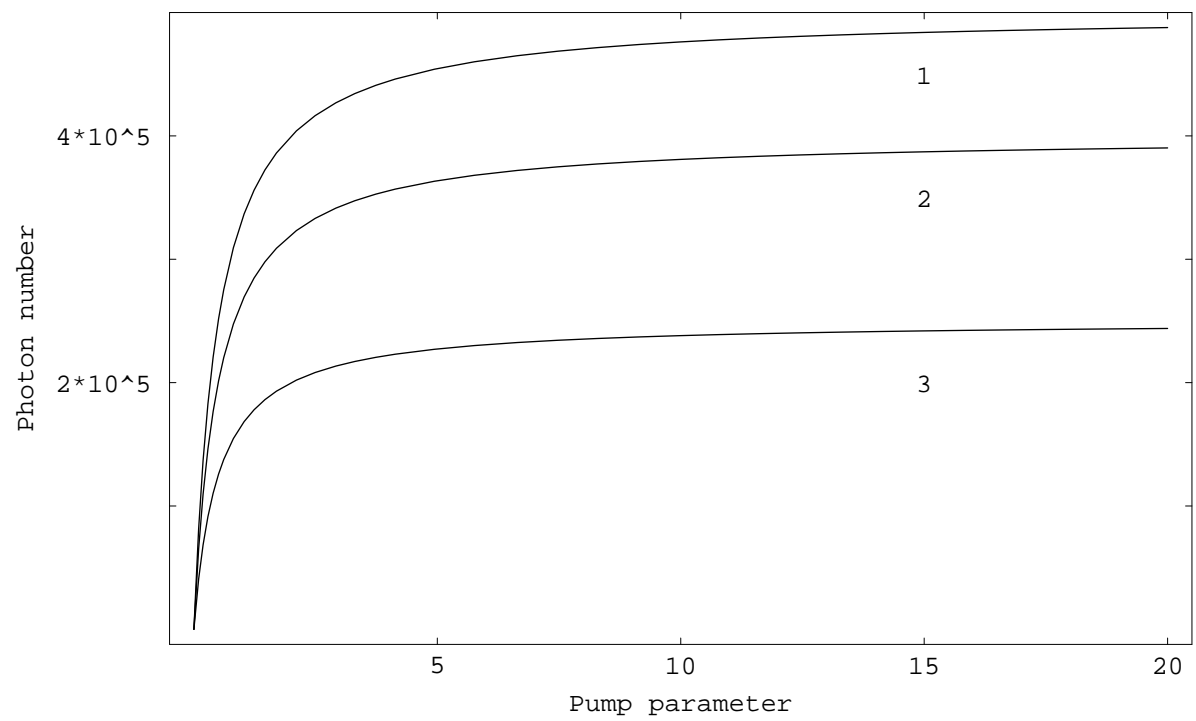


Figure captions

1. Two level scheme.
2. The photon number vs relative pump rate for the two level scheme. The parameters are $\lambda = 10^3$, $\delta = 10^5$, $s = 7 \cdot 10^{-7}$, 10^{-6} and $1.33 \cdot 10^{-6}$ for curves 1, 2 and 3, respectively.
3. Three level schemes with different pump mechanism.
4. Photon number vs pump rate for the pump schemes shown in figs. 3a and 3b. The parameters are: $\lambda_1 = 10^6$, $\epsilon_1 = 0.01$, $\delta_1 = 0$, $s_1 = 0$ (curve 1), 0.2 (curve 2) and 0.5 (curve 3) for fig 4a; $\lambda_2 = 10^5$, $\epsilon_2 = 0$, $\delta_2 = 0.1$, $s_2 = 0.1$ (curve 1), 0.02 (curve 2) and 0.01 (curve 3) for fig 4b.